

RIESZ TYPE POTENTIAL OPERATORS IN GENERALIZED GRAND MORREY SPACES

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ABSTRACT. In this paper we introduce generalized grand Morrey spaces in the framework of quasimetric measure spaces, in the spirit of the so-called grand Lebesgue spaces. We prove a kind of *reduction lemma* which is applicable to a variety of operators to reduce their boundedness in generalized grand Morrey spaces to the corresponding boundedness in Morrey spaces, as a result of this application, we obtain the boundedness of the Hardy-Littlewood maximal operator as well as the boundedness of Calderón-Zygmund potential type operators. Boundedness of Riesz type potential operators are also obtained in the framework of homogeneous and also in the nonhomogeneous case in generalized grand Morrey spaces.

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1. INTRODUCTION

In 1992 T. Iwaniec and C. Sbordone [16], in their studies related with the integrability properties of the Jacobian in a bounded open set Ω , introduced a new type of function spaces $L^{p) }(\Omega)$, called *grand Lebesgue spaces*. A generalized version of them, $L^{p),\theta}(\Omega)$ appeared in L. Greco, T. Iwaniec and C. Sbordone [15]. Harmonic

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analysis related to these spaces and their associate spaces (called *small Lebesgue spaces*), was intensively studied during last years due to various applications, we mention e.g. [2, 7, 9, 10, 11, 12, 18]. Recently in [29] there was introduced a version of weighted grand Lebesgue spaces adjusted for sets $\Omega \subseteq \mathbb{R}^n$ of infinite measure, where the integrability of $|f(x)|^{p-\varepsilon}$ at infinity was controlled by means of a weight, and there generalized grand Lebesgue spaces were also considered, together with the study of classical operators of harmonic analysis in such spaces. Another idea of introducing “bilateral” grand Lebesgue spaces on sets of infinite measure was suggested in [23], where the structure of such spaces was investigated, not operators; the spaces in [23] are two parametrical with respect to the exponent p , with the norm involving $\sup_{p_1 < p < p_2}$.

Morrey spaces $L^{p,\lambda}$ were introduced in 1938 by C. Morrey [24] in relation to regularity problems of solutions to partial differential equations, and provided a useful tool in the regularity theory of PDE’s (for Morrey spaces we refer to books [14, 22], see also [28] where an overview of various generalizations may be found).

Recently, in the spirit of grand Lebesgue spaces, A. Meskhi [25, 26] introduced *grand Morrey spaces* (in [25] it was already defined on quasi-metric measure spaces with doubling measure) and obtained results on the boundedness of the maximal operator, Calderón-Zygmund singular operators and Riesz potentials. The boundedness of commutators of singular and potential operators in grand Morrey spaces was already treated by X. Ye [31]. Note that the “*grandification* procedure” was applied only to the parameter p .

In this paper we make a further step and apply the “grandification procedure” to both the parameters, p and λ , obtaining *generalized grand Morrey spaces* $L_{\theta,A}^{(p),\lambda}(X,\mu)$. In this new framework we obtain a reduction boundedness theorem, which reduces the boundedness of operators (not necessarily linear ones) in generalized grand Morrey spaces to the corresponding boundedness in classical Morrey spaces.

In our future investigations we plan to establish the boundedness of commutators of singular and fractional integrals and its applications, for example, in regularity problems for the solution of elliptic equations in non-divergence form from generalized Morrey spaces viewpoint.

Notation:

d_X denotes the diameter of the X set;

$A \sim B$ for positive A and B means that there exists $c > 0$ such that $c^{-1}A \leq B \leq cA$;

$B(x, r) = \{y \in X : d(x, y) < r\}$;

by c and C we denote various absolute positive constants, which may have different values even in the same line;

\hookrightarrow means continuous imbedding;

f_B denotes the integral average of f , i.e. $f_B := \frac{1}{\mu B} \int_B f \, d\mu$;

p' stands for the conjugate exponent $1/p + 1/p' = 1$.

2. PRELIMINARIES

2.1. Spaces of homogeneous type. Let $X := (X, d, \mu)$ be a topological space with a complete measure μ such that the space of compactly supported continuous functions is dense in $L^1(X, \mu)$ and d is a quasimetric, i.e. it is a non-negative real-valued function d on $X \times X$ which satisfies the conditions:

- (i) $d(x, y) = 0$ if and only if $x = y$;
- (ii) there exists a constant $C_t > 0$ such that $d(x, y) \leq C_t[d(x, z) + d(z, y)]$, for all $x, y, z \in X$, and
- (iii) there exists a constant $C_s > 0$ such that $d(x, y) \leq C_s \cdot d(y, x)$, for all $x, y \in X$.

Let μ be a positive measure on the σ -algebra of subsets of X which contains the d -balls $B(x, r)$. Everywhere in the sequel we assume that all the balls have a finite measure, that is, $\mu B(x, r) < \infty$ for all $x \in X$ and $r > 0$ and that for every neighborhood V of $x \in X$, there exists $r > 0$ such that $B(x, r) \subset V$.

We say that the measure μ is *lower α -Ahlfors regular*, if

$$(1) \quad \mu B(x, r) \geq cr^\alpha$$

and *upper β -Ahlfors regular* (or, it satisfies the *growth condition of degree β*), if

$$(2) \quad \mu B(x, r) \leq cr^\beta,$$

where $\alpha, \beta, c > 0$ does not depend on x and r . When $\alpha = \beta$, the measure μ is simply called *α -Ahlfors regular*.

The condition

$$(3) \quad \mu B(x, 2r) \leq C_d \cdot \mu B(x, r), \quad C_d > 1$$

on the measure μ with C_d not depending on $x \in X$ and $0 < r < d_X$, is known as the *doubling condition*.

Iterating it, we obtain

$$(4) \quad \frac{\mu B(x, R)}{\mu B(y, r)} \leq C_d \left(\frac{R}{r} \right)^{\log_2 C_d}, \quad 0 < r \leq R$$

for all d -balls $B(x, R)$ and $B(y, r)$ with $B(y, r) \subset B(x, R)$.

The triplet (X, d, μ) , with μ satisfying the doubling condition, is said a *space of homogeneous type*, abbreviated from now on simply as SHT. For some important examples of an SHT we refer e.g. to [5].

From (4) it follows that every homogeneous type space (X, d, μ) with a finite measure is lower $(\log_2 C_d)$ -Ahlfors regular.

2.2. Grand Lebesgue spaces. For $1 < p < \infty$, $\theta > 0$ and $0 < \varepsilon < p - 1$ the *grand Lebesgue space* is the set of measurable functions for which

$$(5) \quad \|f\|_{L^{p, \theta}(X, \mu)} := \sup_{0 < \varepsilon < p-1} \varepsilon^{\frac{\theta}{p-\varepsilon}} \|f\|_{L^{p-\varepsilon}(X, \mu)} < \infty$$

where $\|f\|_{L^p(X, \mu)}^p := \int_X |f(y)|^p d\mu(y)$. In the case $\theta = 1$, we denote $L^{p, \theta}(X, \mu) := L^p(X, \mu)$.

When $\mu X < \infty$, then for all $0 < \varepsilon \leq p - 1$ we have

$$L^p(X, \mu) \hookrightarrow L^p(X, \mu) \hookrightarrow L^{p-\varepsilon}(X, \mu).$$

For more properties of grand Lebesgue spaces, see [18].

2.3. Morrey spaces. For $1 \leq p < \infty$ and $0 \leq \lambda < 1$, the usual Morrey space $L^{p,\lambda}(X, \mu)$ is introduced as the set of all measurable functions such that

$$(6) \quad \|f\|_{L^{p,\lambda}(X,\mu)} := \sup_{\substack{x \in X \\ 0 < r < d_X}} \left(\frac{1}{\mu B(x, r)^\lambda} \int_{B(x, r)} |f(y)|^p d\mu(y) \right)^{\frac{1}{p}} < \infty.$$

Sometimes we will need a modification of the previous Morrey space, namely, we define $\mathcal{L}^{p,\lambda}(X, \mu)$ as

$$(7) \quad \|f\|_{\mathcal{L}^{p,\lambda}(X,\mu)} := \sup_{\substack{x \in X \\ 0 < r < d_X}} \left(\frac{1}{r^{\gamma\lambda}} \int_{B(x, r)} |f(y)|^p d\mu(y) \right)^{\frac{1}{p}} < \infty.$$

3. GENERALIZED GRAND MORREY SPACES AND THE REDUCTION LEMMA

In this section we will assume that the measure μ is upper γ -Ahlfors regular. After introducing generalized grand Morrey spaces in the framework of SHT in a slightly more general way as was done for the Euclidean case in H. Rafeiro [27], we show that the same reduction lemma is valid in the setting of SHT.

We introduce the following functional

$$(8) \quad \Phi_{\varphi, A}^{p,\lambda}(f, s) := \sup_{0 < \varepsilon < s} \varphi(\varepsilon)^{\frac{1}{p-\varepsilon}} \|f\|_{L^{p-\varepsilon, \lambda-A(\varepsilon)}(X, \mu)}.$$

Definition 3.1 (Generalized grand Morrey spaces). Let $1 < p < \infty$, $0 \leq \lambda < 1$, φ be a positive bounded function with $\lim_{t \rightarrow 0+} \varphi(t) = 0$ and A be a non-decreasing real-valued non-negative function with $\lim_{x \rightarrow 0+} A(x) = 0$. By $L_{\varphi, A}^{(p), \lambda}(X, \mu)$ we denote the space of measurable functions having the finite norm

$$(9) \quad \|f\|_{L_{\varphi, A}^{(p), \lambda}(X)} := \Phi_{\varphi, A}^{p,\lambda}(f, s_{\max}), \quad s_{\max} = \min\{p-1, a\}$$

where $a = \sup\{x > 0 : A(x) \leq \lambda\}$.

Remark 3.2. For appropriate φ , in the case $A \equiv 0, \lambda > 0$ we recover the Grand Morrey spaces introduced in A. Meskhi [26], and when $\lambda = 0, A \equiv 0$ we have the grand Lebesgue spaces introduced in [15] (and in [16] in the case $\theta = 1$).

For fixed $p, \lambda, \varphi, A, f$ we have that $s \mapsto \Phi_{\varphi, A}^{p,\lambda}(f, s)$ is a non-decreasing function, but it is possible to estimate $\Phi_{\varphi, A}^{p,\lambda}(f, s)$ via $\Phi_{\varphi, A}^{p,\lambda}(f, \sigma)$ with $\sigma < s$ as follows.

Lemma 3.3. For $0 < \sigma < s < s_{\max}$ we have that

$$(10) \quad \Phi_{\varphi, A}^{p,\lambda}(f, s) \leq C \varphi(\sigma)^{-\frac{1}{p-\sigma}} \Phi_{\varphi, A}^{p,\lambda}(f, \sigma),$$

where C depends on γ , the parameters p, λ, φ, A and the diameter d_X , but does not depend on f, s and σ .

Proof. For fixed σ and $0 < \sigma < s < s_{\max}$ we have

$$(11) \quad \Phi_{\varphi, A}^{p,\lambda}(f, s) = \max \left\{ \Phi_{\varphi, A}^{p,\lambda}(f, \sigma), \underbrace{\sup_{\sigma \leq \varepsilon < s} \varphi(\varepsilon)^{\frac{1}{p-\varepsilon}} \|f\|_{L^{p-\varepsilon, \lambda-A(\varepsilon)}(X)}}_{=: I} \right\}.$$

To estimate

$$I = \sup_{\sigma \leq \varepsilon < s} \varphi(\varepsilon)^{\frac{1}{p-\varepsilon}} \sup_{\substack{x \in X \\ 0 < r \leq d_X}} \mu B(x, r)^{\frac{A(\varepsilon)-\lambda}{p-\varepsilon}} \|f\|_{L^{p-\varepsilon}(B(x, r))},$$

since φ is a positive bounded function and by Holder's inequality, we have

$$\begin{aligned}
I &\leq C \sup_{0 < \eta < s_{\max}} \varphi(\eta)^{\frac{1}{p-\eta}} \sup_{\sigma < \varepsilon < s_{\max}} \sup_{\substack{x \in X \\ 0 < r \leq d_X}} \mu B(x, r)^{\frac{1+A(\varepsilon)-\lambda}{p-\varepsilon}} \times \\
&\quad \left(\int_{B(x, r)} |f(y)|^{p-\sigma} d\mu(y) \right)^{\frac{1}{p-\sigma}} \\
&\leq C \sup_{\sigma < \varepsilon < s_{\max}} \sup_{\substack{x \in X \\ 0 < r \leq d_X}} \mu B(x, r)^{\Delta(\varepsilon, \sigma)} \times \\
&\quad \left(\frac{\varphi(\sigma) \varphi(\sigma)^{-1}}{\mu B(x, r)^{\lambda-A(\sigma)}} \int_{B(x, r)} |f(y)|^{p-\sigma} d\mu(y) \right)^{\frac{1}{p-\sigma}}
\end{aligned}$$

with $\Delta(\varepsilon, \sigma) := \frac{1+A(\varepsilon)-\lambda}{p-\varepsilon} - \frac{1+A(\sigma)-\lambda}{p-\sigma}$.

Observe that, since A is non-decreasing we have that

$$\begin{aligned}
\Delta(\varepsilon, \sigma) &= \frac{1+A(\varepsilon)-\lambda}{p-\varepsilon} - \frac{1+A(\sigma)-\lambda}{p-\sigma} \\
&= \frac{(\varepsilon-\sigma)(1-\lambda) + A(\varepsilon)(p-\sigma) - A(\sigma)(p-\varepsilon)}{(p-\sigma)(p-\varepsilon)} \geq 0,
\end{aligned}$$

and for $0 \leq \varepsilon < s_{\max}$ we have $\frac{1-\lambda}{p} \leq \frac{1+A(\varepsilon)-\lambda}{p-\varepsilon} \leq 1$, so that $0 \leq \Delta(\varepsilon, \sigma) \leq 1$. Then

$$\mu B(x, r)^{\frac{1+A(\varepsilon)-\lambda}{p-\varepsilon} - \frac{1+A(\sigma)-\lambda}{p-\sigma}} \leq C \max\{1, d_X^\gamma\}$$

since μ satisfies the γ -growth condition, and we obtain

$$\begin{aligned}
I &\leq C \sup_{\substack{x \in X \\ 0 < r \leq d_X}} \varphi(\sigma)^{-\frac{1}{p-\sigma}} \left(\frac{\varphi(\sigma)}{\mu B(x, r)^{\lambda-A(\sigma)}} \int_{B(x, r)} |f(y)|^{p-\sigma} d\mu(y) \right)^{\frac{1}{p-\sigma}} \\
&\leq C \varphi(\sigma)^{-\frac{1}{p-\sigma}} \sup_{0 < \varepsilon \leq \sigma} \varphi(\varepsilon)^{\frac{1}{p-\varepsilon}} \|f\|_{L^{p-\varepsilon, \lambda-A(\varepsilon)}(X)} \\
&= C \varphi(\sigma)^{-\frac{1}{p-\sigma}} \cdot \Phi_{\varphi, A}^{p, \lambda}(f, \sigma). \quad \square
\end{aligned}$$

From Lemma 3.3 we immediately have

Lemma 3.4. *For $0 < \sigma < s_{\max}$, the norm defined in (9) has the following dominant*

$$(12) \quad \|f\|_{L_{\varphi, A}^{(p), \lambda}(X)} \leq C \frac{\Phi_{\varphi, A}^{p, \lambda}(f, \sigma)}{\varphi(\sigma)^{\frac{1}{p-\sigma}}},$$

Lemma 3.5 (Reduction lemma). *Let U be an operator (not necessarily sublinear) bounded in the Morrey spaces*

$$(13) \quad \|Uf\|_{L^{q-\varepsilon, \lambda-A_2(\varepsilon)}(X)} \leq C_{p-\varepsilon, \lambda-A_1(\varepsilon), q-\varepsilon, \lambda-A_2(\varepsilon)} \|f\|_{L^{p-\varepsilon, \lambda-A_1(\varepsilon)}(X)}$$

for all sufficiently small $\varepsilon \in (0, \sigma]$, where $0 < \sigma < s_{\max}$. If

$$(14) \quad \sup_{0 < \varepsilon < \sigma} C_{p-\varepsilon, \lambda-A_1(\varepsilon), q-\varepsilon, \lambda-A_2(\varepsilon)} < \infty$$

and

$$(15) \quad \sup_{0 < \varepsilon < \sigma} \frac{\psi(\varepsilon)^{\frac{1}{q-\varepsilon}}}{\varphi(\varepsilon)^{\frac{1}{p-\varepsilon}}} < \infty,$$

then it is also bounded in the generalized grand Morrey space

$$(16) \quad \|Uf\|_{L_{\psi, A_2}^{(q), \lambda}(X)} \leq C \|f\|_{L_{\varphi, A_1}^{(p), \lambda}(X)}$$

with

$$C = \frac{C_0}{\varphi(\sigma)^{\frac{1}{p-\sigma}}} \sup_{0 < \varepsilon < \sigma} C_{p-\varepsilon, \lambda-A_1(\varepsilon), q-\varepsilon, \lambda-A_2(\varepsilon)},$$

where C_0 may depend on $\gamma, p, \lambda, \varphi, A$ and d_X , but does not depend on σ and f .

Proof. By (12), we have

$$(17) \quad \|Uf\|_{L_{\psi, A_2}^{(q), \lambda}(X)} \leq \frac{C}{\psi(\sigma)^{\frac{1}{q-\sigma}}} \Phi_{\psi, A_2}^{q, \lambda}(Uf, \sigma).$$

The estimation of $\Phi_{\psi, A_2}^{q, \lambda}(Uf, \sigma)$ by $\|f\|_{L_{\varphi, A_1}^{(p), \lambda}(X)}$ is direct:

$$(18) \quad \begin{aligned} \Phi_{\psi, A_2}^{q, \lambda}(Uf, \sigma) &= \sup_{0 < \varepsilon \leq \sigma} \psi(\varepsilon)^{\frac{1}{q-\varepsilon}} \|Uf\|_{L^{q-\varepsilon, \lambda-A_2(\varepsilon)}(X)} \\ &\leq C \sup_{0 < \varepsilon \leq \sigma} \varphi(\varepsilon)^{\frac{1}{p-\varepsilon}} \cdot C_{p-\varepsilon, \lambda-A_1(\varepsilon), q-\varepsilon, \lambda-A_2(\varepsilon)} \cdot \|f\|_{L^{p-\varepsilon, \lambda-A(\varepsilon)}(X)} \\ &\leq C \sup_{0 < \varepsilon \leq \sigma} C_{p-\varepsilon, \lambda-A_1(\varepsilon), q-\varepsilon, \lambda-A_2(\varepsilon)} \cdot \|f\|_{L_{\varphi, A_1}^{(p), \lambda}(X)} \end{aligned}$$

where the first inequality comes from assumption (15). \square

Remark 3.6. The estimations in Lemmata 3.3, 3.4 and 3.5 are still true, up to a different constant, when we work in the modified Morrey space defined in (7).

3.1. Hardy-Littlewood maximal operator. As an application of the reduction lemma, we obtain the boundedness of the Hardy-Littlewood maximal operator

$$Mf(x) = \sup_{\substack{x \in X \\ 0 < r < d_X}} \frac{1}{\mu B(x, r)} \int_{B(x, r)} |f(y)| d\mu(y)$$

in generalized grand Morrey spaces. The following proposition was shown in A. Meskhi [26]

Proposition 3.7. *Let $1 < p < \infty$ and $0 \leq \lambda < 1$. Then*

$$\|Mf\|_{L^{p, \lambda}(X, \mu)} \leq \left((C_d)^{\frac{\lambda}{p}} c_0(p')^{\frac{1}{p}} + 1 \right) \|f\|_{L^{p, \lambda}(X, \mu)}$$

holds, where the positive constant C_d arises in the doubling condition for μ and c_0 arises from covering lemmas.

Theorem 3.8. *Let $1 < p < \infty$ and $0 \leq \lambda < 1$. Then the Hardy-Littlewood maximal operator is bounded from $L_{\varphi, A}^{(p), \lambda}(X, \mu)$ to $L_{\psi, A}^{(p), \lambda}(X, \mu)$ if exists small σ such that $\sup_{0 < \varepsilon < \sigma} \psi(\varepsilon)^{\frac{1}{q-\varepsilon}} / \varphi(\varepsilon)^{\frac{1}{p-\varepsilon}} < \infty$.*

Proof. The result follows from Lemma 3.5 and by noticing that, from Proposition 3.7 and the definition of generalized grand Morrey space (see Definition 3.1) we have that

$$(C_d)^{\frac{\lambda-A(\varepsilon)}{p-\varepsilon}} c_0((p-\varepsilon)')^{\frac{1}{p-\varepsilon}} < \infty$$

for all $0 < \varepsilon < s_{\max}$, since $\lambda - A(\varepsilon) \geq 0$. \square

3.2. Calderón-Zygmund singular operators. We follow [26] in this section, in particular, making use of the following definition of the Calderón-Zygmund singular operators. Namely, the Calderón-Zygmund operator is defined as the integral operator

$$Tf(x) = \text{p.v.} \int_X K(x, y) f(y) \, d\mu(y)$$

with the kernel $K : X \times X \setminus \{(x, x) : x \in \Omega\} \rightarrow \mathbb{R}$ being a measurable function satisfying the conditions:

$$|K(x, y)| \leq \frac{C}{\mu B(x, d(x, y))}, \quad x, y \in X, \quad x \neq y;$$

$$|K(x_1, y) - K(x_2, y)| + |K(y, x_1) - K(y, x_2)| \leq Cw\left(\frac{d(x_2, x_1)}{d(x_2, y)}\right) \frac{1}{\mu B(x_2, d(x_2, y))}$$

for all x_1, x_2 and y with $d(x_2, y) \geq Cd(x, x_2)$, where w is a positive non-decreasing function on $(0, \infty)$ which satisfies the Δ_2 condition $w(2t) \leq cw(t)$ ($t > 0$) and the Dini condition $\int_0^1 w(t)/t \, dt < \infty$. We also assume that Tf exists almost everywhere on X in the principal value sense for all $f \in L^2(X)$ and that T is bounded in $L^2(X)$. The boundedness of such Calderón-Zygmund operators in Morrey spaces is valid, as can be seen in the following Proposition, proved in [26].

Proposition 3.9. *Let $1 < p < \infty$ and $0 \leq \lambda < 1$. Then*

$$\|Tf\|_{L^{p,\lambda}(X,\mu)} \leq C_{p,\lambda} \|f\|_{L^{p,\lambda}(X,\mu)}$$

where

$$(19) \quad C_{p,\lambda} \leq c \begin{cases} \frac{p}{p-1} + \frac{p}{2-p} + \frac{p-\lambda+1}{1-\lambda} & \text{if } 1 < p < 2, \\ p + \frac{p}{p-2} + \frac{p-\lambda+1}{1-\lambda} & \text{if } p > 2, \end{cases}$$

with c not depending on p and λ .

Theorem 3.10. *Let $1 < p < \infty$, $\theta > 0$, $\alpha \geq 0$ and $0 < \lambda < 1$. Then the Calderón-Zygmund operator T is bounded in generalized grand Morrey spaces $L_{\varphi,A}^{(p),\lambda}(X,\mu)$.*

Proof. Keeping in mind that by Lemma 3.5 we are interested only in small values of ε , from (19), we deduce that

$$C_{p-\varepsilon,\lambda-A(\varepsilon)} \leq c \begin{cases} \frac{p}{p-\varepsilon-1} + \frac{p-\varepsilon}{2-p+\varepsilon} + \frac{p-\varepsilon-\lambda+A(\varepsilon)+1}{1-\lambda+A(\varepsilon)} & \text{if } p \leq 2 \text{ and } 0 < \varepsilon < p-1; \\ p-\varepsilon + \frac{p-\varepsilon}{p-\varepsilon-2} + \frac{p-\varepsilon-\lambda+A(\varepsilon)+1}{1-\lambda+A(\varepsilon)} & \text{if } p > 2 \text{ and } 0 < \varepsilon < p-2. \end{cases}$$

and we are done, since it is possible to choose a small σ such that (14) is valid. \square

4. RIESZ TYPE POTENTIAL OPERATORS IN GENERALIZED GRAND MORREY SPACES

In this section we will assume that the triplet (X, d, μ) is an SHT and the measure μ is upper γ -Ahfors regular.

4.1. Riesz potential operator. By a *Riesz type potential operator*, we mean an operator of the type

$$(20) \quad I^\alpha f(x) := \int_X \frac{f(y)}{d(x, y)^{\gamma-\alpha}} d\mu(y)$$

where $0 < \alpha < \gamma$.

The following proposition was shown in A. Meskhi [26] in the case of Morrey spaces $\mathcal{L}^{q,\lambda}(X, \mu)$ as defined in (7).

Proposition 4.1. *Let $1 < p < \infty$, $0 < \alpha < \frac{(1-\lambda)\gamma}{p}$, $\frac{1}{p} - \frac{1}{q} = \frac{\alpha}{(1-\lambda)\gamma}$, where $0 \leq \lambda < 1$. Then the inequality*

$$(21) \quad \|I^\alpha f\|_{\mathcal{L}^{q,\lambda}(X,\mu)} \leq \bar{c}(p, \alpha, \lambda, \gamma) \|f\|_{\mathcal{L}^{p,\lambda}(X,\mu)}$$

holds, where the positive constant $\bar{c}(p, \alpha, \lambda, \gamma)$ is given by

$$\bar{c}(p, \alpha, \lambda, \gamma) = c \frac{(1-\lambda)\gamma}{\alpha[(1-\lambda)\gamma - \alpha p]} [(p')^{1/q} + 1]$$

and the positive constant c does not depend on p and α .

Using the norm (7), we define the corresponding generalized grand Morrey space, namely

$$(22) \quad \|f\|_{\mathcal{L}_{\varphi,A}^{p,\lambda}(X,\mu)} := \sup_{0 < \varepsilon < s_{\max}} \varphi(\varepsilon)^{\frac{1}{p-\varepsilon}} \|f\|_{\mathcal{L}^{p-\varepsilon,\lambda-A(\varepsilon)}(X,\mu)}$$

where $s_{\max} = \min\{p-1, a\}$ with $a = \sup\{x > 0 : A(x) \leq \lambda\}$. If $\varphi(\varepsilon) := \varepsilon^\theta$, when θ is a positive number, we denote $\|f\|_{\mathcal{L}_{\varphi,A}^{p,\lambda}(X,\mu)} =: \|f\|_{\mathcal{L}_{\theta,A}^{p,\lambda}(X,\mu)}$.

By the Hardy-Littlewood-Sobolev inequality we know that the boundedness of the Riesz potential operator in Lebesgue spaces is valid when the exponents are different and related to each other. In the case of generalized grand Morrey spaces we will have a similar result, but now not only the exponents will be different, also the indices θ_1 and θ_2 will be different.

Before stating and proving the main result in this subsection (Theorem 4.3), we introduce some auxiliary functions those will be used afterwards.

Definition 4.2 (auxiliary functions). On an interval $(0, \delta]$, δ is small, we define the following functions:

$$\bar{\phi}(x) := p + \frac{\gamma(x-q)(1-\lambda+A_2(x))}{\gamma(1-\lambda+A_2(x))-\alpha(x-q)}, \quad \tilde{\phi}(x) := q - \frac{\gamma(p-x)(1-\lambda+A_1(x))}{\gamma(1-\lambda+A_1(x))-\alpha(p-x)}$$

$$\bar{A}(x) = 1 - \frac{\alpha(x-q)}{\gamma(1-\lambda+A_2(x))}, \quad \tilde{A}(x) = \frac{1-\lambda+A_1(\eta)}{\gamma(1-\lambda+A_1(\eta))-(p-\eta)\alpha}$$

$$\phi(x) := \bar{\phi}(x)^{\bar{A}(x)}, \quad \Phi(x) := \tilde{\phi}(x)^{\tilde{A}(x)}$$

$$\psi(\varepsilon) = \phi(\varepsilon^{\theta_1}), \quad \Psi(\varepsilon) = \Phi(\varepsilon^{\theta_1}),$$

for $\theta_1 > 0$.

Theorem 4.3. *Let $1 < p < \infty$, $0 < \alpha < ((1-\lambda)\gamma)/p$, $0 < \lambda < 1$, $1/p - 1/q = \alpha/((1-\lambda)\gamma)$. Suppose that $\theta_1 > 0$ and that $\theta_2 \geq \theta_1[1 + \alpha q/((1-\lambda)\gamma)]$. Let A_1 and A_2 be continuous non-negative functions on $(0, p-1]$ and $(0, q-1]$ respectively satisfying the conditions:*

- (i) $A_2 \in C^1((0, \delta])$ for some positive $\delta > 0$;
- (ii) $\lim_{x \rightarrow 0+} A_2(x) = 0$;
- (iii) $0 \leq B := \lim_{x \rightarrow 0+} \frac{dA_2}{dx}(x) < \frac{(1-\lambda)^2}{\alpha q^2}$;
- (iv) $A_1(\eta) = A_2(\bar{\phi}^{-1}(\eta))$, where $\bar{\phi}^{-1}$ is the inverse of $\bar{\phi}$ on $(0, \delta]$ for some $\delta > 0$.

Then the Riesz potential operator I^α is $(\mathcal{L}_{\theta_1, A_1}^{(p), \lambda}(X, \mu) - \mathcal{L}_{\theta_2, A_2}^{(q), \lambda}(X, \mu))$ -bounded.

Proof. We note that it is enough to prove the theorem for $\theta_2 = \theta_1(1 + \frac{\alpha q}{1-\lambda})$ because $\varepsilon^{\theta_2} \leq \varepsilon^{\theta_1(1 + \frac{\alpha q}{1-\lambda})}$ for $\theta_2 > \theta_1[1 + (\alpha q)/(1-\lambda)]$ and small ε . We also note that, by L'Hospital rule, $\bar{\phi}(x) \sim x$ as $x \rightarrow 0+$ since $B < (1-\lambda)^2/(\alpha q^2)$. Moreover, $\bar{\phi}$ is invertible near 0, since $\frac{d\bar{\phi}}{dx}(x) > 0$. Under the conditions of the Theorem 4.3 the function A_1 is continuous on $(0, \delta]$ and $\lim_{x \rightarrow 0+} A_1(x) = 0$. With all of the previous remarks taken into account, it is enough to prove the boundedness of I^α from $\mathcal{L}_{\theta_1, A_1}^{(p), \lambda}(X, \mu)$ to $\mathcal{L}_{\psi, A_2}^{(q), \lambda}(X, \mu)$ since $\phi(x) \sim x^{1 + \frac{\alpha q}{(1-\lambda)\gamma}}$, and consequently, $\psi(x) = \phi(x^{\theta_1}) \sim x^{\theta_1(1 + \frac{\alpha q}{(1-\lambda)\gamma})}$ as $x \rightarrow 0$. We will now proceed in a similar fashion as in the proof of Lemma 3.3.

The case $\sigma < \varepsilon \leq s_{\max}$, where s_{\max} is from (22). Letting

$$I := \psi^{\frac{1}{q-\varepsilon}}(\varepsilon) \left(\frac{1}{\mu B(x, r)^{\lambda - A_2(\varepsilon)}} \int_{B(x, r)} |I^\alpha f(y)|^{q-\varepsilon} d\mu(y) \right)^{\frac{1}{q-\varepsilon}}$$

we have

$$\begin{aligned} I &\leq \psi^{\frac{1}{q-\varepsilon}}(\varepsilon) \mu B(x, r)^{\frac{A_2(\varepsilon)+1-\lambda}{q-\varepsilon}} \left(\int_{B(x, r)} |I^\alpha f(y)|^{q-\sigma} d\mu(y) \right)^{\frac{1}{q-\sigma}} \\ &\leq C \psi^{\frac{1}{q-\varepsilon}}(\varepsilon) \mu B(x, r)^{\frac{A_2(\sigma)+1-\lambda}{q-\sigma}} \left(\int_{B(x, r)} |I^\alpha f(y)|^{q-\sigma} d\mu(y) \right)^{\frac{1}{q-\sigma}} \\ &\leq C \left(\sup_{\sigma \leq \varepsilon \leq s_{\max}} \psi^{\frac{1}{q-\varepsilon}}(\varepsilon) \right) \psi^{\frac{1}{\sigma-q}}(\sigma) \times \\ &\quad \sup_{0 < \varepsilon \leq \sigma} \sup_{\substack{x \in X \\ r > 0}} \left(\frac{\psi(\varepsilon)}{\mu B(x, r)^{\lambda - A_2(\varepsilon)}} \int_{B(x, r)} |I^\alpha f(y)|^{q-\varepsilon} d\mu(y) \right)^{\frac{1}{q-\varepsilon}} \end{aligned}$$

where the first inequality comes from Hölder's inequality and the second one is due to the fact that A_2 is bounded on $[\sigma, q-1]$ and $x \mapsto (1-\lambda)/(q-x)$ is an increasing function. Hence, it is enough to consider the case $0 < \varepsilon \leq \sigma$.

The case $0 < \varepsilon \leq \sigma$. Let η and ε be chosen so that

$$(23) \quad \frac{1}{p-\eta} - \frac{1}{q-\varepsilon} = \frac{\alpha}{(1-\lambda + A_2(\varepsilon))\gamma}.$$

Obviously we have that $\varepsilon \rightarrow 0$ if and only if $\eta \rightarrow 0$ and solving η with respect to ε in (23) we obtain

$$\eta = p - \frac{\gamma(q-\varepsilon)(1-\lambda + A_2(\varepsilon))}{\gamma(1-\lambda + A_2(\varepsilon)) - \alpha(\varepsilon - q)} = \bar{\phi}(\varepsilon).$$

Letting

$$J := \psi^{\frac{1}{q-\varepsilon}}(\varepsilon) \left(\frac{1}{\mu B(x, r)^{\lambda-A_2(\varepsilon)}} \int_{B(x, r)} |I^\alpha f(y)|^{q-\varepsilon} d\mu(y) \right)^{\frac{1}{q-\varepsilon}}$$

we have

$$\begin{aligned} J &\leq C \frac{(1-\lambda+A_2(\varepsilon))\gamma}{\alpha[(1-\lambda+A_2(\varepsilon))\gamma-\alpha(p-\eta)]} [(p-\eta)']^{\frac{1}{q-\varepsilon}} \psi^{\frac{1}{q-\varepsilon}}(\varepsilon) \times \\ &\quad \sup_{\substack{x \in X \\ r > 0}} \left(\frac{1}{\mu B(x, r)^{\lambda-A_2(\varepsilon)}} \int_{B(x, r)} |f(y)|^{p-\eta} d\mu(y) \right)^{\frac{1}{p-\eta}} \\ &\leq C \frac{(1-\lambda+A_2(\varepsilon))\gamma}{\alpha[(1-\lambda+A_2(\varepsilon))\gamma-\alpha(p-\eta)]} \eta^{\frac{\theta_1}{p-\eta}} \psi^{\frac{1}{q-\varepsilon}}(\varepsilon) \times \\ &\quad \sup_{\substack{x \in X \\ r > 0}} \left(\frac{\eta^{\theta_1}}{\mu B(x, r)^{\lambda-A_2(\varepsilon)}} \int_{B(x, r)} |f(x)|^{p-\eta} d\mu(x) \right)^{\frac{1}{p-\eta}} \\ &\leq C \|f\|_{\mathcal{L}_{\theta_1, A_1}^{(p), \lambda}(X, \mu)} \end{aligned}$$

where the first inequality is due to Proposition 4.1 and the last one is due to the fact that $\eta = \bar{\phi}(\varepsilon)$. Since the constant in the last inequality is uniformly bounded with respect to ε we obtain the desired boundedness of the Riesz potential operator. \square

Corollary 4.4. *Let $1 < p < \infty$, $0 < \alpha < ((1-\lambda)\gamma)/p$, $0 < \lambda < 1$, $1/p - 1/q = \alpha/((1-\lambda)\gamma)$. Suppose that $\theta_1 > 0$ and that $\theta_2 \geq \theta_1(1+\alpha q/(1-\lambda))$. Let $A_2(x) = \alpha x$, where α is a non-negative constant satisfying the condition $\alpha < (1-\lambda)^2/(\alpha q^2)$. Let $A_1(x) = \alpha \bar{\phi}^{-1}(x)$. Then I^α is $(\mathcal{L}_{\theta_1, A_1}^{(p), \lambda}(X, \mu) - \mathcal{L}_{\theta_2, A_2}^{(q), \lambda}(X, \mu))$ -bounded.*

It is also possible to prove a similar result of Theorem 4.3, but now requiring conditions on the function A_1 , namely we have

Theorem 4.5. *Let $1 < p < \infty$, $0 < \alpha < 1$, $0 < \lambda < 1 - \alpha p$, $\frac{1}{p} - \frac{1}{q} = \frac{\alpha}{1-\lambda}$. Suppose that $\theta_1 > 0$ and that $\theta_2 > \theta_1(1 + \frac{\alpha q}{1-\lambda})$. Let A_1 and A_2 be continuous non-negative functions on $(0, p-1]$ and $(0, q-1]$ respectively satisfying the conditions:*

- (i) $A_1 \in C^1((0, \delta])$ for some positive $\delta > 0$;
- (ii) $\lim_{x \rightarrow 0+} A_1(x) = 0$;
- (iii) $B_1 := \lim_{x \rightarrow 0+} \frac{dA_1}{dx}(x) \geq 0$;
- (iv) $A_2(x) = A_1(\bar{\phi}^{-1}(x))$ on $(0, \delta]$ for some $\delta > 0$.

Then the Riesz potential operator I^α is $(\mathcal{L}_{\theta_1, A_1}^{(p), \lambda}(X, \mu) - \mathcal{L}_{\theta_2, A_2}^{(q), \lambda}(X, \mu))$ -bounded.

Proof. The proof is similar to that of Theorem 4.3. In this case it is enough to prove that I^α is $(\mathcal{L}_{\Psi, A_1}^{(p), \lambda}(X, \mu) - \mathcal{L}_{\theta_2, A_2}^{(q), \lambda}(X, \mu))$ -bounded, where $\theta_2 = 1 + \frac{\alpha q}{1-\lambda} = \frac{1-\lambda}{1-\lambda-\alpha p}$. \square

4.2. Riesz potential operator defined via measure. By a *Riesz type potential operator defined via measure*, we mean an operator of the type

$$(24) \quad I_\mu^\alpha f(x) := \int_X \frac{f(y)}{\mu(x, d(x, y))^{1-\alpha}} d\mu(y)$$

where $0 < \alpha < 1$.

The following proposition was proved in A. Meskhi [26].

Proposition 4.6. *Let $1 < p < \infty$, $0 < \alpha < \frac{1-\lambda}{p}$, $\frac{1}{p} - \frac{1}{q} = \frac{\alpha}{1-\lambda}$, where $0 \leq \lambda < 1$. Then the inequality*

$$(25) \quad \|I_\mu^\alpha f\|_{L^{q,\lambda}(X,\mu)} \leq c(p, \alpha, \lambda) \|f\|_{L^{p,\lambda}(X,\mu)}$$

holds, where

$$c(p, \alpha, \lambda) = b_0 \left(C_\alpha + \frac{p}{1-\lambda-\alpha p} \right) [(p')^{1/q} + 1]$$

and the positive constant b_0 does not depend on p and α .

It is also possible to obtain the boundedness of the operator I_μ^α in the framework of generalized grand Morrey spaces, namely the following statement holds:

Theorem 4.7. *Let $1 < p < \infty$, $0 < \alpha < (1-\lambda)/p$, $0 < \lambda < 1$, $1/p - 1/q = \alpha/(1-\lambda)$. Suppose that $\theta_1 > 0$ and that $\theta_2 \geq \theta_1(1 + \alpha q/(1-\lambda))$. Let A_1 and A_2 be continuous non-negative functions on $(0, p-1]$ and $(0, q-1]$ respectively satisfying the conditions:*

- (i) $A_2 \in C^1((0, \delta])$ for some positive $\delta > 0$;
- (ii) $\lim_{x \rightarrow 0+} A_2(x) = 0$;
- (iii) $0 \leq B := \lim_{x \rightarrow 0+} \frac{dA_2}{dx}(x) < \frac{(1-\lambda)^2}{\alpha q^2}$;
- (iv) $A_1(\eta) = A_2(\bar{\phi}^{-1}(\eta))$, where $\bar{\phi}^{-1}$ is the inverse of $\bar{\phi}$ on $(0, \delta]$ for some $\delta > 0$.

Then the Riesz potential operator I^α is $\left(L_{\theta_1, A_1}^{(p), \lambda}(X, \mu) - L_{\theta_2, A_2}^{(q), \lambda}(X, \mu) \right)$ -bounded.

Proof. The proof follows, mutatis mutandis, the proof of Theorem 4.3. Namely, we need to use Proposition 4.6 and the auxiliary functions from Definition 4.2 should be used with $\gamma = 1$. \square

Remark 4.8. If μ is upper Ahlfors regular, then by using the pointwise estimate:

$$I^\alpha f(x) \leq c_\alpha Mf(x), \quad f \geq 0,$$

and Theorem 3.8 we have also the same boundedness for I^α as in Theorem 3.8 for M .

5. POTENTIALS ON NONHOMOGENEOUS SPACES

In this section we will deal with potential operators in the framework of non-homogeneous spaces. Namely, let (X, d, μ) be a topological space with a complete measure μ such that the space of compactly supported functions are dense in $L^1(X, \mu)$ and d is a quasimetric satisfying the standard conditions, see subsection 2.1. As before we will assume that $d_X \equiv \text{diam}(X) < \infty$. In this section, we do NOT assume that μ is doubling!

Let

$$(K_\alpha f)(x) = \int_X \frac{f(y)}{d(x, y)^{1-\alpha}} d\mu(y),$$

where $0 < \alpha < 1$.

We need the following *modified* maximal operator on X

$$(\widetilde{M}f)(x) = \sup_{r>0} \frac{1}{\mu B(x_0, N_0 r)} \int_{B(x, r)} |f(y)| d\mu(y),$$

where $N_0 = C_t(1 + 2C_s)$ and the constants C_s and C_t are from the definition of quasimetric d . Let b be a constant. We will use the symbol bB for a ball $B(x, br)$, where $B \equiv B(x, r)$.

Lemma 5.1. *Let $1 < p < \infty$. Then the following inequality holds for all $f \in L^p(X, \mu)$*

$$(26) \quad \|\widetilde{M}f\|_{L^p(X, \mu)} \leq 2(p')^{\frac{1}{p}} \|f\|_{L^p(X, \mu)}$$

Proof. The operator \widetilde{M} is of weak type $(1, 1)$ with constant 1, i.e. the inequality

$$\mu\left(\left\{x \in X : (\widetilde{M}f)(x) > \lambda\right\}\right) \leq \frac{1}{\lambda} \int_X |f(x)| d\mu(x)$$

holds, see [6, p. 368]. Since \widetilde{M} is of strong type (∞, ∞) with constant 1, i.e. $\|\widetilde{M}f\|_{L^\infty} \leq \|f\|_{L^\infty}$, we conclude that the inequality (26) holds with constant $2(p')^{\frac{1}{p}}$ (see [8, p. 29]). \square

5.1. Modified Morrey space. We will define a modified Morrey space

Definition 5.2 (Modified Morrey space). Let $1 < p < \infty$ and let $0 \leq \lambda < 1$. Suppose that a is a positive constant. We denote by $L^{p, \lambda}(X, \mu)_a$ the modified Morrey space defined by the norm

$$\|f\|_{L^{p, \lambda}(X, \mu)_a} = \sup_{x \in X, r > 0} \left(\frac{1}{\mu B(x, ar)^\lambda} \int_{B(x, r)} |f(y)| d\mu(y) \right)$$

For the next statement we refer to [21], but we give the proof for completeness, because we will need the constant in the inequality.

Lemma 5.3. *Let $1 < p < \infty$ and let $0 \leq \lambda < 1$. Then the following inequality*

$$(27) \quad \|\widetilde{M}f\|_{L^{p, \lambda}(X, \mu)_{N_0 \bar{a}}} \leq \left[1 + 2(p')^{\frac{1}{p}}\right] \|f\|_{L^{p, \lambda}(X, \mu)_{N_0}}$$

holds, where N_0 and \bar{a} are positive constants defined by $N_0 = C_t(1 + 2C_s)$, $\bar{a} = C_t(C_t(C_s + 1) + 1)$.

Proof. Let r be a small positive number and represent f as follows $f = f_1 + f_2$, where $f_1 = f \cdot \chi_{B(x, \bar{a}r)}$, $f_2 = f - f_1$ and \bar{a} is the constant defined above.

We have

$$\begin{aligned} \left[\frac{1}{\mu B(x, N_0 \bar{a}r)^\lambda} \int_{B(x, r)} (\widetilde{M}f)^p(y) d\mu(y) \right]^{\frac{1}{p}} &\leq \\ &\left[\frac{1}{\mu B(x, N_0 \bar{a}r)^\lambda} \int_{B(x, r)} (\widetilde{M}f_1)^p(y) d\mu(y) \right]^{\frac{1}{p}} + \\ &\left[\frac{1}{\mu B(x, N_0 \bar{a}r)^\lambda} \int_{B(x, r)} (\widetilde{M}f_2)^p(y) d\mu(y) \right]^{\frac{1}{p}} =: \\ &J_1(x, r) + J_2(x, r). \end{aligned}$$

By applying Lemma 5.1, we have that

$$\begin{aligned} J_1(x, r) &\leq \frac{1}{\mu B(x, N_0 \bar{a}r)^{\frac{\lambda}{p}}} \left(\int_{B(x, r)} (\widetilde{M}f_1)^p(y) d\mu(y) \right)^{\frac{1}{p}} \\ &\leq 2(p')^{\frac{1}{p}} [\mu B(x, N_0 \bar{a}r)]^{-\frac{\lambda}{p}} \left(\int_{B(x, \bar{a}r)} (f)^p(y) d\mu(y) \right)^{\frac{1}{p}} \\ &\leq 2(p')^{\frac{1}{p}} \|f\|_{L^{p, \lambda}(X, \mu)_{N_0}}. \end{aligned}$$

Observe now that (see also [21, p. 929]) if $y \in B(x, r)$, then $B(x, r) \subset B(y, C_t(C_s + 1)r) \subset B(x, \bar{a}r)$. Hence, for $y \in B(x, r)$

$$(\widetilde{M}f_2)(y) \leq \sup_{B \supset B(x, r)} \frac{1}{\mu(N_0 B)} \int_B |f(z)| d\mu(z).$$

Consequently

$$\begin{aligned} J_2(x, r) &\leq [\mu B(x, N_0 \bar{a}r)]^{-\frac{\lambda}{p}} \sup_{B \supset B(x, r)} \left[\frac{1}{\mu(N_0 B)} \int_B |f(y)| d\mu(y) \right] (\mu B(x, r))^{\frac{1}{p}} \\ &\leq (\mu B(x, N_0 r))^{\frac{1-\lambda}{p}} \sup_{B \supset B(x, r)} \left[\frac{1}{\mu(N_0 B)} \int_B |f(y)|^p d\mu(y) \right]^{\frac{1}{p}} \\ &\leq \sup_B \left[\frac{1}{\mu(N_0 B)^{\frac{\lambda}{p}}} \int_B |f(y)|^p d\mu(y) \right]^{\frac{1}{p}} \\ &= \|f\|_{L^{p, \lambda}(X, \mu)_{N_0}} \end{aligned}$$

Therefore we obtain (27). \square

For the next statement we refer to [17] for Euclidean spaces and [6, p. 367] for nonhomogeneous spaces.

Theorem A. *Let $1 < p < \infty$, $0 < \alpha < 1/p$, $q = p/(1 - \alpha p)$. Then K_α is bounded from $L^p(X, \mu)$ to $L^q(X, \mu)$ if and only if there is a positive constant b such that*

$$(28) \quad \mu B(x, r) \leq br$$

for all $x \in X$ and $r > 0$, i.e., the measure is upper 1-Ahlfors regular.

Remark 5.4. Theorem A is proved for $\mu X = \infty$ but it is also true for $\mu X < \infty$ (observe that $d_X < \infty$ implies $\mu X < \infty$ because $\mu B < \infty$ for all balls!)

Lemma 5.5. *Let the measure μ be upper 1-Ahlfors regular, $1 < p < \infty$, $0 < \alpha < \frac{1-\lambda}{p}$, $0 \leq \lambda < 1$. We set $q = \frac{p(1-\lambda)}{1-\lambda-\alpha p}$. Then the following inequality holds:*

$$\|K_\alpha f\|_{L^{q, \lambda}(X, \mu)_{N_0 \bar{\sigma}}} \leq C_{b, N_0, p, \lambda, \alpha} \|f\|_{L^{p, \lambda}(X, \mu)_{N_0}},$$

where

$$C_{b, N_0, p, \lambda, \alpha} = 4 \left[1 + 2(p')^{\frac{1}{p}} \right]^{\frac{p}{q}} \left[\frac{bN_0}{\alpha} + \frac{b^{\frac{1}{p_1} - \frac{\lambda}{p}} N_0^{\frac{\lambda}{p}} p}{1 - \lambda - \alpha p} \right].$$

with b from (28).

Proof. First we prove the Hedberg's type inequality

$$|K_\alpha f(x)| \leq A_{b,N_0,p,\lambda,\alpha} \widetilde{M}f(x)^{1-\frac{p\alpha}{1-\lambda}} \|f\|_{L^{p,\lambda}(X,\mu)_{N_0}}^{\frac{\alpha p}{1-\lambda}},$$

where

$$A_{b,N_0,p,\lambda,\alpha} = 4 \left[\frac{bN_0}{\alpha} + \frac{b^{\frac{1}{p'} - \frac{\lambda}{p}} N_0^{\frac{\lambda}{p}} p}{1 - \lambda - \alpha p} \right].$$

Observe that the inequality $d(x,y)^{\alpha-1} \leq 2^{2-\alpha} \int_{d(x,y)}^{2d(x,y)} t^{\alpha-2} dt$ holds, where $0 < d(x,y) < d_X$. Hence,

$$\begin{aligned} |K_\alpha f(x)| &\leq 4 \int_X |f(y)| \left(\int_{d(x,y)}^{2d(x,y)} t^{\alpha-2} dt \right) d\mu(y) \\ &= 4 \int_0^{2d_X} t^{\alpha-2} \left(\int_{t/2 < d(x,y) < t} |f(y)| d\mu(y) \right) dt = 4 \left(\int_0^\varepsilon + \int_\varepsilon^{2d_X} \right) \dots \\ &= 4(S_1 + S_2). \end{aligned}$$

By using condition (28) we find that

$$\begin{aligned} S_2 &\leq \int_0^\varepsilon t^{\alpha-1} \left(\frac{1}{t} \int_{B(x,t)} |f(y)| d\mu(y) \right) dt \\ &\leq bN_0 \left(\int_0^\varepsilon t^{\alpha-1} dt \right) \widetilde{M}f(x) \\ &= \frac{bN_0}{\alpha} \varepsilon^\alpha \widetilde{M}f(x), \end{aligned}$$

where b is the constant from (28).

Further, Hölder's inequality and condition (28) yields

$$\begin{aligned} \frac{1}{t} \int_{B(x,t)} |f(y)| d\mu(y) &\leq \frac{(\mu B(x,t))^{\frac{1}{p'}}}{t} \left(\int_{B(x,t)} |f(y)|^p d\mu(y) \right)^{\frac{1}{p}} \\ &\leq \frac{(\mu B(x,t))^{\frac{1}{p'}} (\mu B(x, N_0 t))^{\frac{\lambda}{p}}}{t} \times \\ &\quad \times \left(\frac{1}{(\mu B(x, N_0 t))^\lambda} \int_{B(x,t)} |f(y)|^p d\mu(y) \right)^{\frac{1}{p}} \\ &\leq b^{\frac{1}{p'} + \frac{\lambda}{p}} N_0^{\frac{\lambda}{p}} t^{\frac{\lambda-1}{p}} \|f\|_{L^{p,\lambda}(X,\mu)_{N_0}}. \end{aligned}$$

Hence

$$\begin{aligned} (K_\alpha f)(x) &\leq 4 \left[\frac{bN_0}{\alpha} \varepsilon^\alpha (\widetilde{M}f)(x) + b^{\frac{1}{p'} + \frac{\lambda}{p}} N_0^{\frac{\lambda}{p}} \left(\int_\varepsilon^{2\varepsilon} t^{\frac{\lambda-1}{p} + \alpha - 1} dt \right) \|f\|_{L^{p,\lambda}(X,\mu)_{N_0}} \right] \\ &= 4 \left[\frac{bN_0}{\alpha} \varepsilon^\alpha (\widetilde{M}f)(x) + b^{\frac{1}{p'} + \frac{\lambda}{p}} N_0^{\frac{\lambda}{p}} \frac{\varepsilon^{\frac{\lambda-1}{p} + \alpha}}{\frac{1-\lambda}{p} - \alpha} \|f\|_{L^{p,\lambda}(X,\mu)_{N_0}} \right] \\ &= \left[\frac{bN_0}{\alpha} \varepsilon^\alpha (\widetilde{M}f)(x) + \frac{b^{\frac{1}{p'} - \frac{\lambda}{p}} N_0^{\frac{\lambda}{p}} p}{1 - \lambda - \alpha p} \varepsilon^{\frac{\lambda-1}{p} + \alpha} \|f\|_{L^{p,\lambda}(X,\mu)_{N_0}} \right]. \end{aligned}$$

Let us take

$$\varepsilon = \left[\frac{\|f\|_{L^{p,\lambda}(X,\mu)_{N_0}}}{(\widetilde{M}f)(x)} \right]^{\frac{p}{1-\lambda}}$$

Then

$$(K_\alpha f(x)) \leq 4 \underbrace{\left[\frac{bN_0}{\alpha} + \frac{b^{\frac{1}{p'} - \frac{\lambda}{p}} N_0^{\frac{\lambda}{p}} p}{1 - \lambda - \alpha p} \right]}_{=: A_{b,N_0,p,\lambda,\alpha}} \|f\|_{L^{p,\lambda}(X,\mu)_{N_0}}^{\frac{\alpha-p}{1-\lambda}} (\widetilde{M}f)^{1-\frac{p\alpha}{1-\lambda}}(x).$$

Finally, letting

$$J := \left(\frac{1}{\mu B(x, N_0 \bar{a} r)^\lambda} \int_{B(x,r)} |K_\alpha f(y)|^q d\mu(y) \right)^{\frac{1}{q}}$$

by Lemma 5.3 we have

$$\begin{aligned} J &\leq A_{b,N_0,p,\lambda,\alpha} \|f\|_{L^{p,\lambda}(X,\mu)_{N_0}}^{\frac{\alpha-p}{1-\lambda}} \left[\frac{1}{\mu B(x, N_0 \bar{a} r)^\lambda} \int_{B(x,r)} (\widetilde{M}f(y))^q d\mu(y) \right]^{\frac{1}{q}} \\ &\leq A_{b,N_0,p,\lambda,\alpha} \|f\|_{L^{p,\lambda}(X,\mu)_{N_0}}^{\frac{\alpha-p}{1-\lambda}} \|\widetilde{M}f\|_{L^{p,\lambda}(X,\mu)_{N_0 \bar{a}}}^{\frac{p}{q}} \\ &\leq \left[1 + 2(p')^{\frac{1}{p}} \right]^{\frac{p}{2}} A_{b,N_0,p,\lambda,\sigma} \|f\|_{L^{p,\lambda}(X,\mu)_{N_0}}. \end{aligned} \quad \square$$

Let us study the boundedness of the operator K_α from the space $L_{\theta_1, A_1}^{(p), \lambda}(X, \mu)_{N_0}$ to $L_{\theta_2, A_2}^{(q), \lambda}(X, \mu)_{N_0 \bar{a}}$, where p, q and λ satisfy the conditions of Lemma 5.5 and the space $L_{\theta, A}^{(p), \lambda}(X, \mu)_a$ is defined by the norm

$$\|f\|_{L_{\theta, A}^{(p), \lambda}(X, \mu)_a} = \sup_{0 < \varepsilon \leq p-1} \sup_{\substack{x \in X \\ 0 < r < d_X}} \left[\frac{\varepsilon^\theta}{\mu B(x, ar)^{\lambda - A(\varepsilon)}} \int_{B(x,r)} |f(y)|^{p-\varepsilon} d\mu(y) \right]^{\frac{1}{p-\varepsilon}}.$$

Theorem 5.6. *Let $1 < p < \infty$, $0 < \alpha < (1 - \lambda)/p$, $0 < \lambda < 1$, $1/p - 1/q = \alpha/(1 - \lambda)$. Suppose that $\theta_1 > 0$ and that $\theta_2 \geq \theta_1(1 + \alpha q/(1 - \lambda))$. Let A_1 and A_2 be continuous non-negative functions on $(0, p - 1]$ and $(0, q - 1]$ respectively satisfying the conditions:*

- (i) $A_2 \in C^1((0, \delta])$ for some positive $\delta > 0$;
- (ii) $\lim_{x \rightarrow 0+} A_2(x) = 0$;
- (iii) $0 \leq B := \lim_{x \rightarrow 0+} \frac{dA_2}{dx}(x) < \frac{(1-\lambda)^2}{\alpha q^2}$;
- (iv) $A_1(\eta) = A_2(\bar{\phi}^{-1}(\eta))$, where $\bar{\phi}^{-1}$ is the inverse of $\bar{\phi}$ on $(0, \delta]$ for some $\delta > 0$.

Then the potential operator K^α is $(L_{\theta_1, A_1}^{(p), \lambda}(X, \mu)_{N_0} - L_{\theta_2, A_2}^{(q), \lambda}(X, \mu)_{N_0 \bar{a}})$ -bounded.

Proof. The proof follows the same lines as Theorem 4.3. \square

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